

Polarization tensor of charged gluons in color magnetic background field at finite temperature

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Abstract

We calculate the polarization tensor of charged gluons in a Abelian homogeneous magnetic background field at finite temperature in one loop order Lorentz background field gauge in full generality. Thereby we first determine the ten independent tensor structures. For the calculation of the corresponding form factors we use the Schwinger representation and represent form factors as double parametric integrals and a sum resulting from the Matsubara formalism used. The integrands are given explicitly in terms of hyperbolic trigonometric functions. Like in the case of neutral gluons, the polarization tensor is not transversal. Out of the tensor structures, seven are transversal and three are not. The nontransversal part follows explicitly from our calculations.

1 Introduction

Investigations of non-Abelian gauge fields at finite temperature which started many years ago have been initiated by two fundamental phenomena, the electroweak phase transition and the deconfinement phase transition happening at high temperature. Numerous methods of calculations were developed and applied

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and important results on the phase transitions and properties of the fermion and gauge boson plasma have been obtained. The key element of most calculation schemes is the polarization tensor. It gives information about the spectra of fields and it is necessary to calculate various thermodynamic potentials beyond tree level. This object was investigated in both, continuum quantum field theory and lattice Monte Carlo calculations. In continuum theory, in particular, there is the necessity of resummations of perturbation series because of infrared singularities appearing in higher orders. That was implemented in different resummation schemes such as hard thermal loop resummations [1], [2, 3], nonperturbative solution of the Schwinger-Dyson equations [4]-[10], 2PI effective action [11, 12], etc. A wide bibliography on these problems is adduced in the review [13].

In the past decade it was realized in both, continuum field theory [21, 25, 23, 24] and Monte Carlo computations on a lattice [27, 28] that at high temperature magnetic fields of the order $gB \sim g^4 T^2$, where g is the gauge coupling constant, B is the field strength T is the temperature, are spontaneously created. This phenomenon is important for both high temperature QCD and the early universe where strong magnetic fields of different kinds had to be present [26]. Hence, a comprehensive investigation of non-Abelian gauge fields in a magnetic background at high temperature is in order. Although such investigations also started many years ago, they mainly dealt with the one-loop calculations of the effective potential [14]-[19] with the goal to study the influence of temperature on the instability of the background field related to the tachyonic mode ($n = -1$) which is present in the charged gluon's spectrum $\epsilon_n^2 = p_3^2 + (2n + 1)gB$, $n = -1, 0, 1, \dots$, where p_3 is the momentum component along the field direction. At finite temperature, the situation with the instability is different because a temperature and field dependent gluon magnetic mass can be generated and possibly remove the instability from the spectrum. At present, this problem is not finally settled although in some approximation it has been considered in Refs. [19, 22, 23]. In fact, only few attention was spent to study the radiation spectra of gluons or W -bosons. Frequently, only fragments were considered like the projection to the neutral sector or to tree level states or with special gauge choices [22], [33]. In addition, the question about the transversality was not finally settled what caused a number of problems over the years (see a recent discussion in [20]).

In our previous papers [29, 30] we investigated the pure one-loop gluon polarization tensor in a homogeneous magnetic background field, first at zero and later at finite temperature. In [29] we calculated the polarization tensor for both, the color neutral and the color charged gluons, in covariant gauge at zero temperature. First we determined all possible tensor structure. It turned out that the polarization tensor in the magnetic field is not transversal (although the 'weak' transversality, i.e., if multiplied on both side with the momentum, holds). The corresponding form factors were represented within the Schwinger formalism as double parametric integrals with explicitly known integrands. The nontransversal structure was confirmed by an investigation of the corresponding Slavnov-Taylor

identities [30]. In Ref. [31] the neutral polarization tensor was calculated at finite temperature. The additional tensor structures appearing due to the temperature were determined and all form factors were represented again in terms of the double parametric integrals and an additional sum because of the temperature. In the present paper we make the final step in this series and calculate the charged polarization tensor at finite temperature.

We are faced with two problems. First, we need to find all tensor structures $T_{\mu\nu}$ with the property that only $p_\mu T_{\mu\nu} p_\nu = 0$ holds where p_μ is a non-commuting momentum because of the magnetic field (for details see below), which to our knowledge had not yet been written down in a complete form for nonzero temperature. The second problem is to represent the form factors in a way expressing the weak transversality in an explicit form. For this it was necessary to find in the parametric integrals the necessary structures which allowed to integrate by parts. This is done and the remarkable property observed earlier that the surface terms from the integration by parts cancel just the contributions from the tadpole graphs up to the Debye mass term is confirmed in this case too.

In the following we follow as close as possible our previous paper [31]. In section 2 the basic notations are introduced and the tensor structures of the polarization tensor are described. In section 3 the calculation of the form factors is actually carried out. For instance, in section 3.5 the Debye mass of charged gluons in the magnetic field is obtained. The Conclusion summarizes the main results, the explicit expressions for the one-loop form factors and some discussion. The procedure of restoring the form factors from the expressions calculated in the main text is given in the Appendix.

Throughout the paper we put $\hbar = c = k_B = g = 1$ and also the magnetic field B is put equal to one in the most part. The dependence on B can be restored by $T \rightarrow T/B^{1/2}$ for the temperature and $p_\mu \rightarrow p_\mu/B^{1/2}$ for the momenta.

2 Basic notations

We start from the operator structures $T_{\lambda\lambda'}^{(i)}$ which are allowed by the weak transversality condition,

$$p_\lambda T_{\lambda\lambda'}^{(i)} p_{\lambda'} = 0, \quad (1)$$

which follows from the corresponding relation the polarization tensor has to obey. Here p_λ is the momentum depending on the external field. These structures appear in the expansion in terms of form factors which will be given below. In the magnetic background field, the polarization tensor can be constructed out of the vectors l_μ , h_μ and d_μ ,

$$l_\mu = \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_4 \end{pmatrix}, \quad h_\mu = \begin{pmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix}, \quad d_\mu = \begin{pmatrix} p_2 \\ -p_1 \\ 0 \\ 0 \end{pmatrix}, \quad (2)$$

where the third vector is $d_\mu = F_{\mu\nu}p_\nu$ and we note $p_\lambda = l_\lambda + h_\lambda$, and the matrixes

$$\delta_{\mu\lambda}^{\parallel} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta_{\mu\lambda}^{\perp} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_{\mu\lambda} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

Hence the operator structures can also be constructed out of these quantities only. We mention that with our choice $B = 1$, $F_{\mu\nu}$ in (3) is the field strength of the background field.

Further, the structures $T_{\lambda\lambda'}^i$ can be at most quadratic in the momenta and following the polarization tensor they must be Hermite. Writing down all allowed combinations, a set of linear independent ones is

$$\begin{aligned} T_{\lambda\lambda'}^{(1)} &= l^2 \delta_{\lambda\lambda'}^{\parallel} - l_\lambda l_{\lambda'} \\ T_{\lambda\lambda'}^{(2)} &= h^2 \delta_{\lambda\lambda'}^{\perp} + 2i F_{\lambda\lambda'} - h_\lambda h_{\lambda'} = d_\lambda d_{\lambda'} + i F_{\lambda\lambda'} \\ T_{\lambda\lambda'}^{(3)} &= h^2 \delta_{\lambda\lambda'}^{\parallel} + l^2 \delta_{\lambda\lambda'}^{\perp} - l_\lambda h_{\lambda'} - h_\lambda l_{\lambda'} \\ T_{\lambda\lambda'}^{(4)} &= i(l_\lambda d_{\lambda'} - d_\lambda l_{\lambda'}) + il^2 F_{\lambda\lambda'} - \delta_{\lambda\lambda'}^{\parallel} \\ T_{\lambda\lambda'}^{(5)} &= h^2 \delta_{\lambda\lambda'}^{\parallel} - l^2 \delta_{\lambda\lambda'}^{\perp} \\ T_{\lambda\lambda'}^{(6a)} &= \delta_{\lambda\lambda'}^{\parallel} + l^2 i F_{\lambda\lambda'}, \quad T_{\lambda\lambda'}^{(6b)} = 3\delta_{\lambda\lambda'}^{\perp} + h^2 i F_{\lambda\lambda'}, \end{aligned} \quad (4)$$

where also the identity $i(d_\lambda h_{\lambda'} - h_\lambda d_{\lambda'}) = ih^2 F_{\lambda\lambda'} + \delta_{\lambda\lambda'}^{\perp}$ was used. In our previous paper [29] we mentioned instead of the two structures $T_{\lambda\lambda'}^{(6a)}$ and $T_{\lambda\lambda'}^{(6b)}$ only their sum,

$$T_{\lambda\lambda'}^{(6)} = T_{\lambda\lambda'}^{(6a)} + T_{\lambda\lambda'}^{(6b)}. \quad (5)$$

In the end it will turn out that for the considered one-loop contribution this is sufficient. However, for the calculations in this section it is convenient to keep temporarily separately both, $T_{\lambda\lambda'}^{(6a)}$ and $T_{\lambda\lambda'}^{(6b)}$.

Another remark on the properties of the structures $T_{\lambda\lambda'}^{(i)}$ is that the first four are transversal, $p_\lambda T_{\lambda\lambda'}^{(i)} = T_{\lambda\lambda'}^{(i)} p_{\lambda'} = 0$ holds for $i = 1, 2, 3, 4$ in addition to (1). The first three structures are just a decomposition of the kernel of the quadratic part of the action, Eq. (24) in [29],

$$T_{\lambda\lambda'}^{(1)} + T_{\lambda\lambda'}^{(2)} + T_{\lambda\lambda'}^{(3)} = K_{\lambda\lambda'}(p). \quad (6)$$

In the case of finite temperature, in addition to the magnetic field, we have to account also for the vector u_λ which describes the speed of the heat bath. In the following we assume it to be orthogonal to h_λ . In fact we use $u_\lambda = (0, 0, 0, 1)$. With this vector additional tensor structures obeying the weak transversality condition (1) can be constructed,

$$T_{\lambda\lambda'}^7 = (u_\lambda l_{\lambda'} + l_\lambda u_{\lambda'}) (up) - \delta_{\lambda\lambda'}^{\parallel} (up)^2 - u_\lambda u_{\lambda'} l^2,$$

$$\begin{aligned}
T_{\lambda\lambda'}^8 &= (u_\lambda h_{\lambda'} + h_\lambda u_{\lambda'}) (up) - \delta_{\lambda\lambda'}^\perp (up)^2 - u_\lambda u_{\lambda'} h^2 , \\
T_{\lambda\lambda'}^9 &= u_\lambda id_{\lambda'} - id_\lambda u_{\lambda'} + 2iF_{\lambda\lambda'}(up) , \\
T_{\lambda\lambda'}^{10a} &= \delta_{\lambda\lambda'}^\parallel (up)^2 - u_\lambda u_{\lambda'} l^2, \quad T_{\lambda\lambda'}^{10b} = \delta_{\lambda\lambda'}^\perp (up)^2 - u_\lambda u_{\lambda'} h^2 .
\end{aligned} \tag{7}$$

These structures are linear independent. Below, however, it will turn out that T^{10a} and T^{10b} appear at the one-loop level which we consider here only in intermediate steps and drop out from the final result.

In addition to (7) there exists a further structure,

$$T_{\lambda\lambda'}^D = u_\lambda u_{\lambda'} \tag{8}$$

which fulfills (1) for $p_4 = 0$ (in an obvious, trivial way). Having in mind that the condition (1) holds only if the external moment p of the polarization tensor fits into the Matsubara formalism, i.e., if it is given by $p_4 = 2\pi Tl$ (l integer) than (8) makes sense as a structure being present for $l = 0$ only (or, formally, being proportional to $\delta_{l,0}$). It is just this structure which delivers the Debye mass term. It must be mentioned that $T_{\lambda\lambda'}^D$ is contained as special case also in $T_{\lambda\lambda'}^8$, $T_{\lambda\lambda'}^9$, $T_{\lambda\lambda'}^{10a}$ or in $T_{\lambda\lambda'}^{10b}$ but due to its exceptional role it makes sense to keep it as separate contribution.

Before writing down the expansion of the polarization tensor in terms of form factors we mention one property which follows directly from the basic commutator relation the momentum p_μ obeys, namely $p_\lambda p^2 = (p^2 \delta_{\lambda\lambda'} + 2iF_{\lambda\lambda'}) p_{\lambda'}$. As a consequence, for a function of p^2 the relations

$$\begin{aligned}
p_\lambda f(p^2) &= f(p^2 + 2iF)_{\lambda\lambda'} p_{\lambda'} , \\
f(p^2) p_\lambda &= p_{\lambda'} f(p^2 + 2iF)_{\lambda'\lambda}
\end{aligned} \tag{9}$$

hold where now f must be viewed as a function of a matrix so that it itself becomes a matrix carrying the indices λ and λ' . The same is true with h^2 in place of p^2 .

The decomposition of the polarization tensor can be written in the form

$$\Pi_{\lambda\lambda'}(p) = \sum_i \Pi^{(i)}(l^2, h^2 + 2iF)_{\lambda\lambda'} T_{\lambda''\lambda'}^{(i)} + \Pi^D T_{\lambda\lambda'}^D. \tag{10}$$

The sum includes in general all structures $T_{\lambda\lambda'}^{(i)}$ defined in (4) and in (7). The form factors $\Pi^{(i)}(l^2, h^2 + 2iF)_{\lambda\lambda'}$ depend on l^2 and h^2 only (besides their dependence on the matrices in (3)). In (10) the form factors can be placed also on the right from the operator structures applying both relations (9).

3 Calculation of the polarization tensor

The basic Feynman graph for the polarization tensor is shown in Fig.5 in [29] and the notations of the vertex factors in Fig.2 there. The analytic expression in

momentum space reads

$$\begin{aligned}\Pi_{\lambda\lambda'}(p) &= \int \frac{dk}{(2\pi)^4} \{ \Gamma_{\lambda\nu\rho} G_{\nu\nu'}(p-k) \Gamma_{\lambda'\nu'\rho'} G_{\rho\rho'}(k) \\ &\quad + (p-k)_\lambda G(p-k) k_{\lambda'} G(k) + k_\lambda G(p-k) (p-k)_{\lambda'} G(k) \} \\ &\quad + \Pi_{\lambda\lambda'}^{\text{tadpol}},\end{aligned}\tag{11}$$

where the second line results from the ghost contribution and the tadpole contribution is given by

$$\begin{aligned}\Pi_{\lambda\lambda'}^{\text{tadpol}} &= \int \frac{dk}{(2\pi)^4} \{ \delta_{\lambda\lambda'} G_{\rho\rho}(k) - G(k)_{\lambda\lambda'} \} \\ &\quad + \int \frac{dp'}{(2\pi)^4} \{ \delta_{\lambda\lambda'} G_{\rho\rho}(p') + G_{\lambda'\lambda}(p') - 2G_{\lambda\lambda'}(p') \}.\end{aligned}\tag{12}$$

The vertex factor is given by

$$\Gamma_{\lambda\nu\rho} = (k-2p)_\rho \delta_{\lambda\nu} + \delta_{\rho\nu}(p-2k)_\lambda + \delta_{\rho\lambda}(p+k)_\nu.\tag{13}$$

For the following we rearrange it in the form

$$\begin{aligned}\Gamma_{\lambda\nu\rho} &= \underbrace{(p-2k)_\lambda \delta_{\nu\rho}}_{\Gamma_\lambda^{(1)}} + \underbrace{2(p_\nu \delta_{\lambda\rho} - p_\rho \delta_{\lambda\nu})}_{\Gamma_{\lambda\nu\rho}^{(2)}} + \underbrace{(-(p-k)_\nu \delta_{\lambda\rho} + k_\rho \delta_{\lambda\nu})}_{\Gamma_{\lambda\nu\rho}^{(3)}}, \\ &\equiv \Gamma_\lambda^{(1)} + \Gamma_{\lambda\nu\rho}^{(2)} + \Gamma_{\lambda\nu\rho}^{(3)},\end{aligned}\tag{14}$$

where $\Gamma_{\lambda\nu\rho}^{(3)}$ will be temporary further subdivided into two parts,

$$\Gamma_{\lambda\nu\rho}^{(3_1)} = -(p-k)_\nu \delta_{\lambda\rho} \quad \text{and} \quad \Gamma_{\lambda\nu\rho}^{(3_2)} = k_\rho \delta_{\lambda\nu}.\tag{15}$$

The momentum integration in the polarization tensor is carried out using the formalism introduced by Schwinger, [32]. There the propagators (in Feynman gauge, $\xi = 1$) are represented as parametric integrals,

$$G(p-k) = \int_0^\infty ds e^{-s(p-k)^2}, \quad G(k) = \int_0^\infty dt e^{-tk^2}\tag{16}$$

for the scalar lines and

$$\begin{aligned}G_{\nu\nu'}(p-k) &= \int_0^\infty ds e^{-s(p-k)^2} E_{\nu\nu'} \\ G_{\rho\rho'}(k) &= \int_0^\infty dt e^{-tk^2} \delta_{\rho\rho'}\end{aligned}\tag{17}$$

for the vector lines with

$$E_{\nu\nu'} \equiv e^{-2isF} = \delta_{\lambda\lambda'}^{\parallel} - iF_{\lambda\lambda'} \sinh(2s) + \delta_{\lambda\lambda'}^{\perp} \cosh(2s).\tag{18}$$

In this formalism the momentum integration is written as an averaging procedure in some auxiliary space and for the basic exponential

$$\hat{\Theta} = e^{-s(p-k)^2} e^{-tk^2} \quad (19)$$

it holds

$$\int \frac{dk}{(2\pi)^4} \hat{\Theta} = \langle \hat{\Theta} \rangle = \Theta(l^2, h^2) \quad (20)$$

with

$$\Theta(l^2, h^2) = \frac{\exp[-H]}{(4\pi)^2(s+t)\sqrt{\Delta}}, \quad (21)$$

which is the result of the corresponding calculations, see [32] for details. The following notations are used,

$$H = \frac{st}{s+t} l^2 + m(s, t) h^2 \quad (22)$$

and

$$\begin{aligned} m(s, t) &\equiv s - \operatorname{arctanh} \frac{p}{q} \\ &= \frac{1}{2} \ln \frac{1 + 2t - e^{-2s}}{1 - (1 - 2t)e^{-2s}} \end{aligned} \quad (23)$$

as well as

$$\begin{aligned} \Delta &= (q^2 - p^2) / 4 \\ &= t^2 + t \sinh(2s) + p/2 \end{aligned} \quad (24)$$

with the notations

$$p = \cosh(2s) - 1, \quad (25)$$

$$q = 2t + \sinh(2s), \quad (26)$$

which will be met frequently in the following. With these notations the self energy graph for scalar lines becomes represented by the parametric integrals in the form

$$\Pi_{(T=0)}^{\text{scalar}} = \int \frac{dk}{(2\pi)^4} G(p-k)G(k) = \int ds dt \Theta(l^2, h^2). \quad (27)$$

These formulas are derived for $T = 0$. To include nonzero temperature, within the Matsubara formalism we are using we have to substitute the integration over the continuous momentum k_4 by a discrete sum over l in $k_4 = 2\pi l T$,

$$\int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \rightarrow T \sum_{l=-\infty}^{\infty}. \quad (28)$$

In order to incorporate this into the parametric integral we represent

$$\begin{aligned} T \sum_{l=-\infty}^{\infty} &= T \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_4 \delta(k_4 - 2\pi l T) \\ &= T \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma e^{-i\sigma 2\pi l T} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} e^{i\sigma k_4} \end{aligned} \quad (29)$$

in this way keeping the original formalism on the expense of accommodating the additional factor $\exp(i\sigma k_4)$ and carrying out the integration over σ and the summation over l afterwards. Under the above assumption that the speed of the heat bath is orthogonal to h_λ , the additional factor $\exp(i\sigma k_4)$ can be incorporated into Schwinger's formalism quite trivially because the integration over k_4 (in the same way as that over k_3) decouples from the other ones in the sense that the corresponding integrals factorize. For the integration over k_4 we have

$$\int_{-\infty}^{\infty} \frac{dk_4}{2\pi} e^{i\sigma k_4} e^{-s(p_4 - k_4)^2 - tk_4^2} = \frac{\exp\left(-\frac{\sigma^2}{4(s+t)} + i\frac{\sigma s p_4}{s+t} - \frac{st}{s+t} p_4^2\right)}{\sqrt{4\pi(s+t)}} \quad (30)$$

and

$$\begin{aligned} T \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} e^{-\sigma k_4} e^{-s(p_4 - k_4)^2 - tk_4^2} \\ = T \sum_{l=-\infty}^{\infty} \exp\left(-(2\pi l T)^2 + 4\pi l T s(up) - \frac{s^2}{s+t}(up)^2 - \frac{st}{s+t} p_4^2\right). \end{aligned} \quad (31)$$

In this formula we used $(up) = p_4$ because this form of writing will be more convenient below.

Combining these with (20), (21) and (20) we get for a graph consisting of scalar lines

$$\begin{aligned} \Pi^{\text{scalar}} &\equiv T \sum_{l=-\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} G(p-k) G(k) \\ &= T \sum_{l=-\infty}^{\infty} \int ds dt e^{-(2\pi l T)^2(s+t) + 4\pi l T s(up) - \frac{s^2}{s+t}(up)^2} \sqrt{4\pi(s+t)} \Theta(l^2, h^2). \end{aligned} \quad (32)$$

This representation is still not in a form which is sufficiently convenient for the following, for instance, it contains still the ultraviolet divergence which appears from small s , t and large l . Using the well known resummation formula

$$\sum_l \exp(-zl^2 + al) = \sqrt{\frac{\pi}{z}} \sum_N \exp\left(-\frac{\pi^2 N^2}{z} + i\pi N \frac{a}{z} + \frac{a^2}{4z^2}\right) \quad (33)$$

(both sums run over the integers) we obtain with $z \rightarrow (2\pi T)^2$, $a \rightarrow 4\pi T(up)$

$$\Pi^{\text{scalar}} = \sum_N \int ds dt \Theta_T(l^2, h^2). \quad (34)$$

Here we introduced the basic average

$$\Theta_T(l^2, h^2) = \exp \left\{ -\frac{N^2}{4(s+t)T^2} + 2s(\tilde{u}p) \right\} \Theta(l^2, h^2) , \quad (35)$$

where $\Theta(l^2, h^2)$ given by Eq.(21) and where the notation

$$\tilde{u}_\lambda = \frac{iN}{2(s+t)T} u_\lambda \quad (36)$$

was introduced. This average is what comes at finite temperature in place of (20),

$$T \sum_{l=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} = \langle \hat{\Theta} \rangle_T = \Theta_T(l^2, h^2) . \quad (37)$$

In (35) the ultraviolet divergence is in the $N = 0$ contribution taken at $B = 0$. The well known basic properties of the representations as sum over l or as sum over N are that the sum over l is convenient for high temperature ($l = 0$ – the so called static mode, gives the leading contribution for $T \rightarrow \infty$) and the sum over N gives the low temperature expansion, for instance the $N = 0$ term is the contribution at $T = 0$.

At this place it is meaningful to show how the magnetic field can be restored in the expressions involving the parameters s and t . Since in the proper time representation of the Green's functions (16), (17) the phase is dimensionless and since the restoration of the B -dependence has to respect that, we must take $s(p-k)^2 \rightarrow (sB)((p-k)^2)/B$. Hence in the parameters s and t the magnetic field is restored by $s \rightarrow sB$ and $t \rightarrow tB$.

We continue with the remark that since below $\Theta(l^2, h^2)$ will become part of the form factors it is meaningful to write it as a function of $h^2 + 2iF$. This can be done by means of the relation

$$\Theta(l^2, h^2) = \Theta(l^2, h^2 + 2iF)Z \quad (38)$$

with

$$Z = -E^\top \frac{D}{D^\top} = \delta^\parallel + \frac{\alpha}{4N} iF + \frac{\beta}{4N} \delta^\perp , \quad (39)$$

where the notations

$$\begin{aligned} \alpha &= (p^2 + q^2) \sinh(2s) - 2pq \cosh(2s), \\ \beta &= (p^2 + q^2) \cosh(2s) - 2pq \sinh(2s), \end{aligned} \quad (40)$$

and

$$A = E - 1 , \quad D = A - 2itF \quad (41)$$

are introduced.

From Schwinger's formalism we need also the commutator relation

$$p_\mu \Theta(l^2, h^2) = \Theta(l^2, h^2) \bar{p}_\mu \quad (42)$$

with

$$\bar{p}_\mu = (Ep)_\mu - (Ak)_\mu. \quad (43)$$

Here we used obvious short notations like $(Ep)_\mu = E_{\mu\mu'} p_{\mu'}$. Finally, we need the average formulas for vectors,

$$\begin{aligned} \langle \hat{\Theta} k_\lambda \rangle_T &= \langle \hat{\Theta} \rangle_T \left(\frac{A}{D} p + \tilde{u} \right)_\lambda, \\ \langle \hat{\Theta} k_\lambda k_{\lambda'} \rangle_T &= \langle \Theta \rangle_T \left[\left(\frac{A}{D} p + \tilde{u} \right)_\lambda \left(\frac{A}{D} p + \tilde{u} \right)_{\lambda'} + \left(\frac{iF}{D^\top} \right)_{\lambda\lambda'} \right], \end{aligned} \quad (44)$$

together with the explicit representation

$$\frac{A}{D} = \frac{s}{s+t} \delta^\parallel - \frac{tp}{2N} iF + \frac{p+t \sinh(2s)}{2N} \delta^\perp. \quad (45)$$

All other quantities can be calculated from these, for example,

$$\frac{iF}{D^\top} = \frac{1}{2} \left(\frac{-2iF}{D} \right)^\top = \frac{1}{2t} \left(1 - \frac{A}{D} \right)^\top = \frac{1}{2} \left(\frac{\delta^\parallel}{s+t} + \frac{p}{2N} iF + \frac{q}{2N} \delta^\perp \right). \quad (46)$$

Perhaps it is useful to remark that all these matrices commute one with the other, that the transposition changes the sign of F and that the simple algebra $\delta^{\parallel 2} = \delta^\parallel$, $\delta^{\perp 2} = \delta^\perp$, $F^2 = -\delta^\perp$, $\delta^\parallel \delta^\perp = \delta^\parallel F = 0$ and $\delta^\perp F = F$ holds.

The averages in (44) are calculated at $T \neq 0$. For $T = 0$ they reduce to the formulas known from [32]. For $T \neq 0$ one needs to consider the corresponding generalizations of Eq.(30),

$$\int_{-\infty}^{\infty} \frac{dk_4}{2\pi} k_4 e^{i\sigma k_4} e^{-s(p_4 - k_4)^2 - tk_4^2}. \quad (47)$$

Replacing the additional factor k_4 by $i \frac{\partial}{\partial \sigma}$, after integration over σ an additional factor $2\pi lT$ appears in a formula which generalized Eq.(32). It remains to do the resummation from l to N . Taking the derivative with respect to a from Eq.(33) after some calculations the first line in (44) for $\lambda = 4$ appears. The derivation of (44) is then finished by the remark that for $\lambda = 1, 2, 3$ no additional contributions appear. In a similar way also the second line in Eq.(44) can be derived.

Now we turn to the calculation of the polarization tensor (11). Using (17), (19), (20) and (42) it can be written in the form

$$\begin{aligned} \Pi_{\lambda\lambda'} &= \sum_N \int ds dt \langle \hat{\Theta}_T [\Gamma_{\lambda\nu\rho} E_{\nu\nu'} \Gamma_{\lambda'\nu'\rho} + (\bar{p} - k)_\lambda k_{\lambda'} + k_\lambda (p - k)_{\lambda'}] \rangle \\ &+ \Pi_{\lambda\lambda'}^{\text{tadpol}}, \end{aligned} \quad (48)$$

where in $\Gamma_{\lambda\nu\rho}$ one needs to substitute p by \bar{p} .

At the next step we divide the whole expression into parts according to the division made in (14) and (15),

$$\Pi_{\lambda\lambda'} = \sum_N \int ds dt \langle \hat{\Theta}_T \left[\sum_{i,j} \hat{M}_{\lambda\lambda'}^{i,j} + \hat{M}_{\lambda\lambda'}^{\text{gh}} \right] \rangle + \Pi_{\lambda\lambda'}^{\text{tadpol}} \quad (49)$$

with

$$\hat{M}_{\lambda\lambda'}^{i,j} = \Gamma_{\lambda\nu\rho}^{(i)} E_{\nu\nu'} \Gamma_{\lambda'\nu'\rho}^{(j)} \quad (50)$$

and $\hat{M}_{\lambda\lambda'}^{\text{gh}}$ is the corresponding contribution from the ghost loop.

The sums in (49) include also the decomposition (15). The explicit expressions for these quantities read

$$\begin{aligned} \hat{M}^{1,1} &= (\bar{p} - 2k)_\lambda (p - 2k)_{\lambda'} \text{tr} E, \\ \hat{M}^{1,2} &= 2 (\bar{p} - 2k)_\lambda \left((E - E^\top) p \right)_{\lambda'}, \\ \hat{M}^{1,3_1} &= -(\bar{p} - 2k)_\lambda (E(p - k))_{\lambda'}, \\ \hat{M}^{1,3_2} &= Z (\bar{p} - 2k)_\lambda (E^\top k)_{\lambda'}, \\ \hat{M}^{2,1} &= 2 \left((E^\top - E) \bar{p} \right)_\lambda (p - 2k)_{\lambda'}, \\ \hat{M}^{2,2} &= 4\delta_{\lambda\lambda'} (\bar{p} E p) - 4 \left(E^\top \bar{p} \right)_{\lambda'} p_\lambda \\ &\quad + 4 (Z)_{\lambda\lambda'} (\bar{p} p) - 4 (\bar{p})_{\lambda'} (E p)_\lambda, \\ \hat{M}^{2,3_1} &= -2\delta_{\lambda\lambda'} (\bar{p} E (p - k)) + 2\bar{p}_{\lambda'} (E(p - k))_\lambda, \\ \hat{M}^{2,3_2} &= -2 (E)_{\lambda\lambda'} (\bar{p} k) + 2 (E^\top \bar{p})_{\lambda'} k_\lambda, \\ \hat{M}^{3_1,1} &= - \left(E^\top (\bar{p} - k) \right)_\lambda (p - 2k)_{\lambda'}, \\ \hat{M}^{3_2,1} &= (E k)_\lambda (p - 2k)_{\lambda'}, \\ \hat{M}^{3_1,2} &= -2\delta_{\lambda\lambda'} ((\bar{p} - k) E p) + 2 \left(E^\top (\bar{p} - k) \right)_{\lambda'} p_\lambda, \\ \hat{M}^{3_2,2} &= -2 (E)_{\lambda\lambda'} (k p) + 2 k_{\lambda'} (E p)_\lambda, \\ \hat{M}^{3,3} &= \delta_{\lambda\lambda'} ((\bar{p} - k) E (p - k)) + E_{\lambda\lambda'} k^2 - k_\lambda \left(E^\top (\bar{p} - k) \right)_{\lambda'} - (E(p - k))_\lambda k_{\lambda'}, \\ \hat{M}^{\text{gh}} &= (\bar{p} - k)_\lambda k_{\lambda'} + k_\lambda (p - k)_{\lambda'}. \end{aligned} \quad (51)$$

Using Eq. (43) and simple relations like $A + 2 = E + 1$ and $EE^\top = 1$ these expressions can be simplified further. After that we apply the average formulas (44) and pass from $\Theta(l^2, h^2)$ to $\Theta(l^2, h^2 + 2iF)$ by means of (38) which brings a factor Z to the $M^{i,j}$ which couples to the index λ , in detail, $ZM^{i,j}$ stands for $Z_{\lambda\lambda'} M_{\lambda''\lambda'}^{i,j}$. In this way we come to

$$\langle \hat{\Theta} \hat{M}^{i,j} \rangle_T = \Theta_T(l^2, h^2 + 2iF) M^{i,j} \quad (52)$$

with

$$M^{1,1} = \left((Pp - 2\tilde{u})_\lambda \left(P^\top p - 2\tilde{u} \right)_{\lambda'} + 2Z \frac{2iF}{D^\top} \right) \text{tr} E$$

$$\begin{aligned}
M^{1,2} &= (Pp - 2\tilde{u})_\lambda (Rp)_{\lambda'}, \\
M^{1,3_1} &= (Pp - 2\tilde{u})_\lambda (-Q_{31}p + \tilde{u})_{\lambda'} - ZE^\top \frac{2iF}{D^\top} \\
M^{1,3_2} &= (Pp - 2\tilde{u})_\lambda (Q_{32}p + \tilde{u})_{\lambda'} - ZE \frac{2iF}{D^\top} \\
M^{2,1} &= (R^\top p)_\lambda (P^\top p - 2\tilde{u})_{\lambda'}, \\
M^{2,2} &= -4 \left[(Sp)_\lambda (Tp)_{\lambda'} + (T^\top p)_\lambda (S^\top p)_{\lambda'} - S_{\lambda\lambda'} (pT^\top p) - T_{\lambda\lambda'}^\top (pSp) \right] \\
&\quad + 4ST^\top 2iF, \\
M^{2,3_1} &= 2(V^\top p - \tilde{u})_\lambda (S^\top p)_{\lambda'} - 2Z_{\lambda\lambda'} \left(\left(p \frac{2itF}{D^\top} p \right) - (\tilde{u}p) \right) - 2iFZ \frac{2itF}{D^\top}, \\
M^{2,3_2} &= 2(Up + \tilde{u})_\lambda (Tp)_{\lambda'} - 2(EZ)_{\lambda\lambda'} \left(\left(p \left(\frac{A}{D} \right)^\top p \right) + (\tilde{u}p) \right) + ZA \frac{2iF}{D^\top}, \\
M^{3_1,1} &= (-Q_{31}^\top p + \tilde{u})_\lambda (P^\top p - 2\tilde{u})_{\lambda'} - ZE^\top \frac{2iF}{D^\top}, \\
M^{3_2,1} &= (Q_{32}^\top p + \tilde{u})_\lambda (P^\top p - 2\tilde{u})_{\lambda'} - ZE \frac{2iF}{D^\top}, \\
M^{3_1,2} &= 2(Sp)_\lambda (Vp - \tilde{u})_{\lambda'} - 2Z_{\lambda\lambda'} \left(\left(p \frac{2itF}{D^\top} p \right) - (\tilde{u}p) \right) - 2iFZ \frac{2itF}{D^\top}, \\
M^{3_2,2} &= 2(T^\top p)_\lambda (U^\top p + \tilde{u})_{\lambda'} - 2(EZ)_{\lambda\lambda'} \left(\left(p \left(\frac{A}{D} \right)^\top p \right) + (\tilde{u}p) \right) + ZA \frac{2iF}{D^\top}, \\
M^{3,3} + M^{\text{gh}} &= Z_{\lambda\lambda} \left(\left(\left(\left(1 - \frac{A}{D} \right) p - \tilde{u} \right) \left(\left(1 - \frac{A}{D} \right) p - \tilde{u} \right) \right) - \text{tr} E \frac{iF}{D^\top} \right) \\
&\quad + (EZ)_{\lambda\lambda'} \left(\left(\frac{A}{D} p + \tilde{u} \right) \left(\frac{A}{D} p + \tilde{u} \right) + \text{tr} \frac{iF}{D^\top} \right),
\end{aligned} \tag{53}$$

where in the last line some cancellations occurred. Here, again, new notations were introduced, namely

$$\begin{aligned}
P &= \left(1 - 2\frac{A}{D} \right)^\top = Z \left(E - (E+1)\frac{A}{D} \right), \\
&= -\frac{s-t}{s+t} \delta^\parallel - \frac{tp}{\Delta} iF - \frac{p/2 - t^2}{\Delta} \delta^\perp \equiv r_3 \delta^\parallel + \alpha_3 iF + \beta_3 \delta^\perp,
\end{aligned}$$

$$R = 2(E - E^\top) = -4 \sinh(2s) iF,$$

$$\begin{aligned}
Q_{31} &= \left(Z \left(1 - \frac{A}{D} \right) \right)^\top = E \frac{-2itF}{D}, \\
&= \frac{t}{s+t} \delta^\parallel + t \frac{p \cosh(2s) - q \sinh(2s)}{2\Delta} iF + t \frac{q \cosh(2s) - p \sinh(2s)}{2\Delta} \delta^\perp, \\
&\equiv s_4 \delta^\parallel + \gamma_4 iF + \delta_4 \delta^\perp,
\end{aligned}$$

$$\begin{aligned}
Q_{32} &= \left(EZ \frac{A}{D} \right)^\top = E^\top \frac{A}{D}, \\
&= \frac{s}{s+t} \delta^\parallel + \frac{p(\sinh(2s) + t)}{2\Delta} iF + \frac{p \cosh(2s) + t \sinh(2s)}{2\Delta} \delta^\perp, \\
&\equiv s_3 \delta^\parallel + \gamma_3 iF + \delta_3 \delta^\perp, \\
S &= Z = \left(E - A \frac{A}{D} \right)^\top = \delta^\parallel + \frac{\alpha}{4\Delta} iF + \frac{\beta}{4\Delta} \delta^\perp, \\
&\equiv r_1 \delta^\parallel + \alpha_1 iF + \beta_1 \delta^\perp, \\
T &= (EZ)^\top = 1 + A^\top \frac{A}{D} = \delta^\parallel + \frac{2pq}{4\Delta} iF + \frac{p^2 + q^2}{4\Delta} \delta^\perp, \\
&\equiv s_2 \delta^\parallel + \gamma_2 iF + \delta_2 \delta^\perp, \\
U &= \left(\frac{A}{D} \right)^\top = \frac{s}{s+t} \delta^\parallel + \frac{tp}{2\Delta} iF + \frac{p + t \sinh(2s)}{2\Delta} \delta^\perp, \\
&\equiv r_2 \delta^\parallel + \alpha_2 iF + \beta_2 \delta^\perp, \\
V &= 1 - \frac{A}{D} = \frac{t}{s+t} \delta^\parallel + \frac{tp}{2\Delta} iF + \frac{tq}{2N} \delta^\perp, \\
&\equiv s_1 \delta^\parallel + \gamma_1 iF + \delta_1 \delta^\perp,
\end{aligned} \tag{54}$$

where (39), (45) and (46) were used. The notations r_i , α_i , β_i , s_i , γ_i and δ_i are introduced here for later use.

As for the dependence on \tilde{u} we made use of the fact that only the fourth component of \tilde{u}_μ is nonzero such that relations like $Z\tilde{u} = S\tilde{u} = T\tilde{u} = \tilde{u}$ and $R\tilde{u} = 0$ hold.

In order to continue and, for instance, to find the necessary structures for integration by parts, we divide the contributions into 3 parts,

1. $M^{1,1} + M^{1,2} + M^{2,1} + M^{2,2}$,
2. $M^{1,3} + M^{2,3} + M^{3,1} + M^{3,2}$
3. $M^{3,3} + M^{\text{gh}}$

and consider them individually in the following subsections.

3.1 Contribution from $M^{1,1} + M^{1,2} + M^{2,1} + M^{2,2}$

We start with $M^{1,1}$. First of all we note that the relation

$$-\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \left(P_{\lambda\lambda} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) = 2Z \frac{2iF}{D^\top} \tag{55}$$

holds which can be verified by differentiation of (39) and (46). It should be mentioned that the term $-2\frac{\tilde{u}_\lambda\tilde{u}_{\lambda'}}{(\tilde{u}p)}$ in the left hand side vanishes under differentiation. It was added by hindsight in order to carry out the integration by parts and to derive a convenient representation for the form factor.

Eq.(55) allows us to represent the contribution of $M^{1,1}$ to the polarization tensor (49) in the form (up to the sum over N)

$$\int dsdt \Theta_T(l^2, h^2 + 2iF) \left(- \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \left(P - 2\frac{\tilde{u}_\lambda\tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \right) \text{tr} E . \quad (56)$$

In this integral we temporarily change variables to λ and u according to $s = \lambda u$, $t = \lambda(1 - u)$ and with $\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) = \frac{1}{\lambda} \frac{\partial}{\partial u}$ we integrate the derivative with respect to u by parts,

$$\begin{aligned} & \int_0^\infty d\lambda \lambda \int_0^1 du \Theta_T(l^2, h^2 + 2iF) \left(-\frac{1}{\lambda} \frac{\partial}{\partial u} \left(P - 2\frac{\tilde{u}_\lambda\tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \right) \text{tr} E \\ = & \int_0^\infty d\lambda \lambda \left[-\Theta_T(l^2, h^2 + 2iF) \left(P - 2\frac{\tilde{u}_\lambda\tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \text{tr} E \right] \Big|_{u=0}^1 \\ & + \int dsdt \Theta_T(l^2, h^2 + 2iF) \left\{ \left(P - 2\frac{\tilde{u}_\lambda\tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \left((p\widetilde{P}p) - 2(\tilde{u}p) \right) \text{tr} E \right. \\ & \left. + 4 \sinh(2s)P \right\} , \quad (57) \end{aligned}$$

where

$$(pPp) = - \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) H - \frac{1}{2\Delta} \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \Delta \quad (58)$$

follows from (22), (24) and (54) and

$$- \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \Theta_T = \Theta_T \left((p\widetilde{P}p) - 2(\tilde{u}p) \right) \quad (59)$$

from (35). We have to take into account that we differentiate a Θ , which depends on $h^2 + 2iF$, so that we get in place of (58)

$$(p\widetilde{P}p) \equiv (pPp) \Big|_{h^2 \rightarrow h^2 + 2iF} . \quad (60)$$

In such expressions we shall use the tilde as notation for this substitution in the following several times. Further we used $\text{tr} E = 2(1 + \cosh(2s))$.

As a result from the integration by parts we get surface contributions which have the general form

$$M_{\text{surface}} = \int_0^\infty d\lambda \lambda \Theta_T(l^2, h^2 + 2iF) \sum_{i,j} M_{\text{sf}}^{i,j} \Big|_{u=0}^1 , \quad (61)$$

The surface contribution from $M^{1,1}$ is then

$$M_{\text{sf}}^{1,1} = - \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \text{tr} E . \quad (62)$$

Taking the contribution from the last line in (57) we represent up to the surface contribution $M^{1,1}$ in the form

$$\begin{aligned} M^{1,1} = & \left((Pp - 2\tilde{u})_\lambda (P^\top p - 2\tilde{u})_{\lambda'} - \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) ((p\widetilde{P}p) - 2(\tilde{u}p)) \right) \text{tr} E \\ & + 4 \sinh(2s) \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) . \end{aligned} \quad (63)$$

Next we consider $M^{1,2} + M^{2,1}$. It is useful to rewrite these in the form

$$\begin{aligned} M^{1,2} + M^{2,1} = & (Pp - 2\tilde{u})_\lambda (Rp)_{\lambda'} + (R^\top p)_\lambda (P^\top p - 2\tilde{u})_{\lambda'} \\ & - \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) (p\widetilde{R^\top}p) - R_{\lambda\lambda'}^\top ((p\widetilde{P}p) - 2(\tilde{u}p)) \\ & + \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) (-4 \sinh(2s)) + R_{\lambda\lambda'}^\top ((p\widetilde{P}p) - 2(\tilde{u}p)) \end{aligned} \quad (64)$$

In the last line we use $(p\widetilde{R^\top}p) = -4 \sinh(2s)$. In fact, by the next-to-last and the last lines we added zero. Now we shall integrate by parts in the last term in the last line. Using the same procedure as before and, for instance, Eq.(59), we get the surface contribution

$$M_{\text{sf}}^{1,2} = -\Theta_T(l^2, h^2 + 2iF) R_{\lambda\lambda'}^\top \quad (65)$$

and using

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) R_{\lambda\lambda'}^\top = 8 \cosh(2s) iF \quad (66)$$

we represent

$$\begin{aligned} M^{1,2} + M^{2,1} = & (Pp - 2\tilde{u})_\lambda (Rp)_{\lambda'} + (R^\top p)_\lambda (P^\top p - 2\tilde{u})_{\lambda'} \\ & - \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) (p\widetilde{R^\top}p) - R_{\lambda\lambda'}^\top ((p\widetilde{P}p) - 2(\tilde{u}p)) \\ & - 4 \sinh(2s) \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) + 8 \cosh(2s) iF_{\lambda\lambda'} . \end{aligned} \quad (67)$$

Finally we turn to $M^{2,2}$. Here we rewrite

$$\begin{aligned} (pSp) &= (p\widetilde{S}p) + (S + S^\top) iF , \\ (pT^\top p) &= (p\widetilde{T^\top}p) + (T + T^\top) iF \end{aligned} \quad (68)$$

and represent $M^{2,2}$ in the form

$$M^{2,2} = -4 \left[(Sp)_\lambda (Tp)_{\lambda'} + (T^\top p)_\lambda (S^\top p)_{\lambda'} - S_{\lambda\lambda'} (p\widetilde{T^\top} p) - T_{\lambda\lambda'}^\top (p\widetilde{Sp}) \right] - 8 \cosh(2s) iF. \quad (69)$$

The last line is the result of a number of cancellations.

We continue with the observation that the last lines in (63), (67) and (69) compensate each other so that we are left with the corresponding first lines. These are written in a form as considered in Appendix A so that we can apply directly the equations (108), (113), (116) and for the temperature part (119) and so on. The first line in Eq.(63) has the form as given by (114) and (115) resp. (126) and (127) multiplied by $\text{tr} E$, the first lines of Eqs.(67) and (69) match the structure of W_3 in (109) resp. (118). The parameters r_i , α_1 , β_i , s_i , γ_i and δ_i are given in Eq.(54).

In this way we get from this subsection the following contribution to the form factors, which we denote by M_i^a :

$$\begin{aligned} M_1^a &= -\left(\frac{s-t}{s+t}\right)^2 2(1 + \cosh(2s)) + 8, \\ M_2^a &= \frac{(tp)^2 - (p/2 - t^2)^2}{\Delta^2} 2(1 + \cosh(2s)) - 8 \sinh(2s) \frac{tp}{\Delta} + 8 \cosh(2s), \\ M_3^a &= \frac{s-t}{s+t} \frac{t^2 - p/2}{\Delta} 2(1 + \cosh(2s)) + \frac{\beta + p^2 + q^2}{\Delta}, \quad (70) \\ M_4^a &= -\frac{s-t}{s+t} \frac{tp}{\Delta} 2(1 + ch) + 4 \sinh(2s) \frac{s-t}{s+t} + \frac{\alpha - 2pq}{\Delta} \end{aligned}$$

(11) (12) + (21) (22)

where in the last line the origin of the contribution is indicated. We note that there are no contributions to M_5 , M_{6a} or M_{6b} .

For the temperature induced contributions we get with Eq.(127) with $\mu = 2\frac{iN}{2(s+t)T}$

$$\begin{aligned} M_7^a &= -2 \frac{iN}{2(s+t)T} \frac{r_3}{(up)} \text{tr} E, \quad M_7^a = -2 \frac{iN}{2(s+t)T} \frac{\beta_3}{(up)} \text{tr} E, \\ M_9^a = -M_{(*)}^a &= 2 \frac{iN}{2(s+t)T} \alpha_3 \text{tr} E. \end{aligned} \quad (71)$$

3.2 Contribution from $M^{1,3} + M^{2,3} + M^{3,1} + M^{3,2}$

In this subsection we employ the subspitting (15). First we consider $M^{31,1} + M^{31,2}$ from (53) and do a reordering using (54),

$$\begin{aligned} M^{31,1} + M^{31,2} &= 2 (Sp)_\lambda (Vp - \tilde{u})_{\lambda'} - 2S_{\lambda\lambda'} \left((pV^\top p) - \tilde{u}p \right) - 2 (V + V^\top) SiF \\ &\quad \left(-Q_{31}^\top p + \tilde{u} \right)_\lambda \left(P^\top p - 2\tilde{u} \right)_{\lambda'} + 2 (V + V^\top) SiF - ZE^\top \frac{2iF}{D^\top} - 2iFZ \frac{2itF}{D^\top} \end{aligned} \quad (72)$$

The reason for the reordering is that the last three terms in the second line collect into a derivative,

$$2 \left(V + V^\top \right) SiF - ZE^\top \frac{2iF}{D^\top} - 2iFZ \frac{2itF}{D^\top} = \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \left(Q_{31}^\top - \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right), \quad (73)$$

a relation which can be checked by explicit taking the derivatives. For instance, the term $-\frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)}$ vanishes under differentiation. It was added by hindsight. In a very similar way we rewrite $M^{1,31} + M^{2,31}$ in the form

$$\begin{aligned} M^{1,31} + M^{2,31} &= 2 \left(V^\top p - \tilde{u} \right)_\lambda \left(S^\top p \right)_{\lambda'} - 2S_{\lambda\lambda'} \left((pV^\top p) - \tilde{u}p \right) - 2 \left(V + V^\top \right) SiF \\ &\quad + (Pp - 2\tilde{u})_\lambda (-Q_{31}p + \tilde{u})_{\lambda'} + \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \left(Q_{31}^\top - \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right). \end{aligned} \quad (74)$$

Now we integrate by parts in the same way as before. The surface contribution is

$$M_{\text{sf}}^2 = 2\Theta_T(l^2, h^2 + 2iF) \left(Q_{31}^\top - \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \quad (75)$$

and in the bulk part we get

$$\begin{aligned} M^{31,1} + M^{31,2} + M^{1,31} + M^{2,31} &= 2 \left\{ (Sp)_\lambda (Vp - \tilde{u})_{\lambda'} + \left(V^\top p - \tilde{u} \right)_\lambda \left(S^\top p \right)_{\lambda'} - 2S_{\lambda\lambda'} \left((p\widetilde{V^\top} p) - \tilde{u}p \right) \right\} \\ &\quad - \left\{ (Pp - 2\tilde{u})_\lambda (Q_{31}p - \tilde{u})_{\lambda'} \right. \\ &\quad \left. + \left(Q_{31}^\top p - \tilde{u} \right)_\lambda \left(P^\top p - 2\tilde{u} \right)_{\lambda'} - 2 \left(Q_{31}^\top - \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \left((p\widetilde{P} p) - \tilde{u}p \right) \right\}, \end{aligned} \quad (76)$$

where also relations like (68) were used. We observe that the two lines in the last formula have just the structure as one of that given in Appendix A. For the upper line it is W_2 with $\mu = 0$ and $\nu = \frac{iN}{2(s+t)T}$ multiplied by an overall factor of 2 and for the lower two lines it is W_1 with $\mu = 2\frac{iN}{2(s+t)T}$ and $\nu = \frac{iN}{2(s+t)T}$ multiplied by an overall factor of (-1) . The corresponding contributions to the form factors can be obtained by means of (108), (113), (119) and (123).

In a similar way the contributions from $M^{32,1} + M^{32,2}$ can be written as

$$\begin{aligned} M^{32,1} + M^{32,2} &= 2 \left(T^\top p \right)_\lambda \left(U^\top p + \tilde{u} \right)_{\lambda'} - 2T_{\lambda\lambda'}^\top ((pUp) + \tilde{u}p) - 2 \left(U + U^\top \right) T^\top iF \\ &\quad + \left(Q_{32}^\top p + \tilde{u} \right)_\lambda \left(P^\top p - 2\tilde{u} \right)_{\lambda'} + 2 \left(U + U^\top \right) T^\top iF - Z \frac{2iF}{D^\top} \end{aligned} \quad (77)$$

and and that from $M^{1,32} + M^{2,32}$ are

$$\begin{aligned} M^{1,32} + M^{2,32} &= 2 (Up + \tilde{u})_\lambda (Tp)_{\lambda'} - 2T_{\lambda\lambda'}^\top ((pUp) + \tilde{u}p) - 2 \left(U + U^\top \right) T^\top iF \\ &\quad + (Pp - 2\tilde{u})_\lambda (Q_{32}p)_{\lambda'} + 2 \left(U + U^\top \right) T^\top iF - Z \frac{2iF}{D^\top}. \end{aligned} \quad (78)$$

In this case the derivative has the form

$$2 \left(U + U^\top \right) T^\top iF - Z \frac{2iF}{D^\top} = - \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \left(Q_{32}^\top + \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \quad (79)$$

where again $\frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)}$ is a term not contributing here but added by hindsight. The resulting surface contribution is

$$M_{\text{sf}}^2 = -2\Theta_T(l^2, h^2 + 2iF) \left(Q_{32}^\top + \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \quad (80)$$

and the corresponding bulk parts after partial integration read

$$\begin{aligned} M^{32,1} + M^{32,2} + M^{1,32} + M^{2,32} \\ = 2 \left\{ (Up + \tilde{u})_\lambda (Tp)_{\lambda'} + (T^\top p)_\lambda (U^\top p + \tilde{u})_{\lambda'} - 2T_{\lambda\lambda'}^\top ((p\widetilde{Up}) + (\tilde{u}p)) \right\} \\ + (Pp - 2\tilde{u})_\lambda (Q_{32}p + \tilde{u})_{\lambda'} + (Q_{32}^\top p + \tilde{u})_\lambda (P^\top p - 2\tilde{u})_{\lambda'} \\ - 2 \left(Q_{32}^\top + \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) ((p\widetilde{Up}) + (\tilde{u}p)). \end{aligned} \quad (81)$$

These have also the form as given in Appendix A, in both with W_1 . In the upper line one has to take $\mu = -\frac{iN}{2(s+t)T}$ and $\nu = 0$ with an overall factor of 2 and in the lower lines $\mu = 2\frac{iN}{2(s+t)T}$ and $\nu = -\frac{iN}{2(s+t)T}$. Their contributions to the form factors can be obtained by Eqs. (108) and (113) and (119) and again we use the notations r_i , α_i , β_i , s_i , γ_i and δ_i taken from (54) and $r_4 = r_3$, $\alpha_4 = \alpha_3$, $\beta_4 = \beta_3$. Further we take into account the numerical prefactors like 2 and -1, and we denote the contributions to the form factors originating from this subsection by M_i^b . These read

$$\begin{aligned} M_2^b &= \frac{1}{2\Delta} \left(-\cosh^2(2s) + (2 - 4t^2) \cosh(2s) - 12t \sinh(2s) \right), \\ M_3^b &= \frac{1}{2(s+t)\Delta} \left(-2t^3 - 10st^2 - 2(t^2 + st + 2) \cosh(2s)t \right. \\ &\quad \left. - 8(s+t) \sinh(2s)t + 5t - (s+t) \cosh^2(2s) + s \right), \\ M_4^b &= \frac{1}{2(s+t)\Delta} \left(4t(t-s) - (s+t) (2t^2 + 1) \sinh(2s) \right. \\ &\quad \left. + \cosh(2s)(4(s-t)t + (s+t) \sinh(2s)) \right), \\ M_5^b &= \frac{1}{2\Delta} \left((\cosh(2s) - 1) (2t^2 + \cosh(2s) - 1) \right), \\ M_{6a}^b &= \frac{1}{2\Delta} \left((-2t^2 + \cosh(2s) - 1) \sinh(2s) \right), \\ M_{6b}^b &= \frac{1}{2\Delta} \left((-2t^2 + \cosh(2s) - 1) \sinh(2s) \right). \end{aligned} \quad (82)$$

We observe that M_{6a}^b and M_{6b}^b are equal. Also, we remind that there was no contribution to M^6 from the preceding subsection and there will be none from the next subsection. As a consequence, the operators structures $T_{\lambda\lambda'}^{(6a)}$ and $T_{\lambda\lambda'}^{(6b)}$ come with the same form factors, hence these contributions collect into the structure $T_{\lambda\lambda'}^{(6)}$, Eq.(5).

Finally we collect the temperature induced part. Its form factors can be written in the form

$$\begin{aligned}
M_7^b &= \frac{iN}{2(s+t)T} \frac{1}{(up)} (-2r_1 + r_4 + 2s_4 + 2s_2 + r_3 - 2s_3) , \\
M_8^b &= \frac{iN}{2(s+t)T} \frac{1}{(up)} (-2\beta_1 + \beta_4 + 2\delta_4 + 2\delta_2 + \beta_3 - 2\delta_3) , \\
M_{10a}^b &= \frac{iN}{2(s+t)T} \frac{1}{(up)} (2r_1 + r_4 - 2s_4 - 2s_2 + r_3 + 2s_3) , \\
M_{10b}^b &= \frac{iN}{2(s+t)T} \frac{1}{(up)} (2\beta_1 + \beta_4 - 2\delta_4 - 2\delta_2 + \beta_3 + 2\delta_3) , \\
M_9^b &= \frac{iN}{2(s+t)T} (2\alpha_1 + \alpha_4 + 2\gamma_4 + 2\gamma_2 - \alpha_3 - 2\gamma_4) , \\
M_{(*)}^b &= \frac{iN}{2(s+t)T} (2\alpha_4 + 2\alpha_3) .
\end{aligned} \tag{83}$$

3.3 Contribution from $M^{3,3} + M^{\text{gh}}$

These contributions need a treatment to some extent different from the preceding two subsections. First of all, we remove the factor Z introduced into $M^{3,3} + M^{\text{gh}}$ in Eq.(53) by means of (38) and we introduce a separate notation for the corresponding contribution to the polarization tensor,

$$\Pi_{\lambda\lambda'}^{33+\text{gh}} = \sum_N \int ds dt \Theta_T(l^2, h^2) M_{\lambda\lambda'}^{33+\text{gh}} \tag{84}$$

with

$$\begin{aligned}
M_{\lambda\lambda'}^{3,3} + M_{\lambda\lambda'}^{\text{gh}} &= \delta_{\lambda\lambda} \left(\left(\left(\left(1 - \frac{A}{D} \right) p - \tilde{u} \right) \left(\left(1 - \frac{A}{D} \right) p - \tilde{u} \right) \right) - \text{tr} E \frac{iF}{D^\top} \right) \\
&\quad + E_{\lambda\lambda'} \left(\left(\frac{A}{D} p + \tilde{u} \right) \left(\frac{A}{D} p + \tilde{u} \right) + \text{tr} \frac{iF}{D^\top} \right) .
\end{aligned} \tag{85}$$

Now we use the statements that the relations

$$\Theta_T(l^2, h^2) \left(\left(\left(1 - \frac{A}{D} \right) p - \tilde{u} \right) \left(\left(1 - \frac{A}{D} \right) p - \tilde{u} \right) \right) = -\frac{\partial}{\partial s} \Theta_T(l^2, h^2) \tag{86}$$

and

$$\Theta_T(l^2, h^2) \left(\left(\frac{A}{D} p + \tilde{u} \right) \left(\frac{A}{D} p + \tilde{u} \right) + \text{tr} \frac{iF}{D^\top} \right) = -\frac{\partial}{\partial t} \Theta_T(l^2, h^2) \tag{87}$$

hold which can be verified by direct differentiation. In this way, in (84) one integration can be carried out and we arrive at

$$\Pi_{\lambda\lambda'}^{33+\text{gh}} = -\sum_N \int_0^\infty ds E_{\lambda\lambda'} \Theta_T(l^2, h^2) \Big|_{t=0}^{t=\infty} - \sum_N \int_0^\infty dt \delta_{\lambda\lambda'} \Theta_T(l^2, h^2) \Big|_{s=0}^{s=\infty} \quad (88)$$

3.4 Tadpole and surface contributions

In this subsection we collect the contributions resulting from the tadpole graphs given by Eq.(12) and the surface contributions which appeared in the preceding subsections.

The tadpole contributions can be calculated easily since they are special cases of the basic loop contribution for $s = 0$ collapsing the line of the charged gluon and keeping the line of the neutral gluon and for $t = 0$ which the lines interchanged. All other rules remain valid so that these contributions can be written down easily,

$$\begin{aligned} \Pi_{\lambda\lambda'}^{\text{tadpol}} = & \sum_N \int dt \Theta_T(l^2, h^2) (-\delta_{\lambda\lambda'} + 4\delta_{\lambda\lambda}) \Big|_{s=0} \\ & + \sum_N \int ds \Theta_T(l^2, h^2) \left(\text{tr} E \delta_{\lambda\lambda'} - 4 \sinh(2s) iF - \left(\delta_{\lambda\lambda'}^{\parallel} + iF \sinh(2s) + \delta_{\lambda\lambda'}^{\perp} \right) \right) \Big|_{t=0}. \end{aligned} \quad (89)$$

Now we collect the surface terms. A part of them has the form (61),

$$M_{\lambda\lambda'}^{\text{surface}} = \sum_N \int_0^\infty d\lambda \Theta_T(l^2, h^2 + 2iF) M_{\lambda\lambda'}^{\text{sf}} \Big|_{u=0}^1 \quad (90)$$

with contributions to $M_{\lambda\lambda'}^{\text{sf}}$ from Eqs.(62), (65), (75) and (80),

$$M_{\lambda\lambda'}^{\text{sf}} = - \left(P_{\lambda\lambda'} - 2 \frac{\tilde{u}_\lambda \tilde{u}_{\lambda'}}{(\tilde{u}p)} \right) \text{tr} E - R_{\lambda\lambda'}^\top + 2Q_{31\lambda\lambda'}^\top - 2Q_{32\lambda\lambda'}^\top. \quad (91)$$

which can be rewritten in the form

$$\Pi_{\lambda\lambda'}^{\text{surface}} = \sum_N \int ds \Theta_T(l^2, h^2) M_{\text{sf}} \Big|_{t=0} - \sum_N \int dt \Theta_T(l^2, h^2) M_{\text{sf}} \Big|_{s=0}. \quad (92)$$

Now it can be checked that when adding (89) and (92) and doing the obvious simplification all contributions cancel except for that which are proportional to $u_\lambda u_{\lambda'}$, i.e., to $T_{\lambda\lambda'}^{\text{D}}$, Eq.(8). These collect into Π^{D} defined in Eq.(10),

$$\Pi^{\text{D}} = \sum_N \int ds \frac{iN}{2s} \Theta_T(l^2, h^2) (-4 + 2\text{tr} E) \Big|_{t=0} - \sum_N \int dt \frac{iN}{2t} \Theta_T(l^2, h^2) (4 - 2\text{tr} E) \Big|_{s=0}. \quad (93)$$

Now we need from (21) and (35)

$$\begin{aligned}\Theta_T|_{s=0} &= \frac{\exp\left(-\frac{N^2}{4tT^2}\right)}{(4\pi)^2 t^2}, \\ \Theta_T|_{t=0} &= \frac{s}{\sinh(s)} \frac{\exp\left(-\frac{N^2}{4sT^2}\right)}{(4\pi)^2 s^2} \exp\left(\frac{iN}{T}p_4\right)\end{aligned}\quad (94)$$

and after renaming the integration variables we obtain for the remaining contributions

$$\begin{aligned}\Pi^D &= \sum_N \int_0^\infty \frac{d\lambda}{\lambda} \frac{iN}{2Tp_4} \left[-4\Theta_T|_{s=0} + 4\Theta_T|_{t=0} \right] \\ &= -\frac{4}{(4\pi)^2} \sum_N \int_0^\infty \frac{d\lambda}{\lambda^3} \frac{iN}{2Tp_4} \left(1 - \frac{\lambda \cosh(2\lambda)}{\sinh(\lambda)} \exp\left(i\frac{Np_4}{T}\right) \right) \exp\left(-\frac{N^2}{4\lambda T^2}\right).\end{aligned}\quad (95)$$

Thus, in our representation, the only contribution coming from the surface terms and tadpoles is the form factor Π^D .

3.5 Debye mass of charged gluons

The expression for Π^D derived in the preceding subsection can be a bit simplified by writing as a sum over $N > 0$ (note the contribution from $N = 0$ vanishes),

$$\Pi^D = -\frac{4}{(4\pi)^2} \sum_{N=1}^\infty \int_0^\infty \frac{d\lambda}{\lambda^3} \frac{\sin(Np_4/T)}{Tp_4/N} \frac{\lambda \cosh(2\lambda)}{\sinh(\lambda)} \exp\left(-\frac{N^2}{4\lambda T^2}\right). \quad (96)$$

It is obvious that for external momenta obeying $p_4 = 2\pi l/T$ (l integer) only the contribution from $l = 0$ is nonzero and we arrive at the Debye mass in the field presence $\Pi^D = -\delta_{l,0} m_D^2$,

$$m_D^2(B) = \frac{1}{4\pi^2} \sum_{N=1}^\infty \int_0^\infty \frac{d\lambda}{\lambda^2} \left(\frac{N}{T}\right)^2 \frac{\cosh(2\lambda)}{\sinh(\lambda)} \exp\left(-\frac{N^2}{4\lambda T^2}\right). \quad (97)$$

This expression coincides with the Debye mass of the neutral gluon, Eq.(123) in [31] that may serve as a check of the correctness of carried out calculations.

The integral over λ is divergent at the upper limit due to the tachyonic mode. In fact, in this contribution one has to written the parametric representation from the very beginning on an axis rotated by 90 degrees clockwise and in the now divergent contributions rotate in anti-Wick direction as this was done in Ref [31]. As a result, in the high temperature limit, $B/T^2 \ll 1$, one obtains

$$\begin{aligned}m_D^2(B) &= \frac{2}{3}T^2 \left[1 - 0.8859 \left(\frac{\sqrt{B}}{2T}\right) + 0.4775 \left(\frac{B^2}{16T^4}\right) \right. \\ &\quad \left. - i 0.4775 \left(\frac{\sqrt{B}}{2T}\right) + O\left(\frac{B^3}{T^6}\right) \right],\end{aligned}\quad (98)$$

where the numeric values of the coefficients are substituted. We see that the real part in the magnetic background is smaller than without the field. The imaginary part is small because it appears in the next-to-leading order $\sim \sqrt{BT}$.

4 Conclusion

Here we collect the contributions which were calculated in the preceding subsections. The form factors appearing in the decomposition (10) of the polarization tensor read

$$\Pi^i(l^2, h^2 + 2iF) = \sum_N \int_0^\infty ds \int_0^\infty dt \Theta_T(l^2, h^2 + 2iF) M_i \quad (99)$$

with

$$\begin{aligned} M_1 &= \frac{4(s+t)^2 - 2(s-t)^2 \cosh(2s)}{(s+t)^2} \\ M_2 &= \frac{1}{\Delta} \left(-2t^2 + 4 \sinh(2s)t + 2 \cosh^2(2s) + \cosh(2s) (4t^2 + 2 \sinh(2s)t - 3) + 1 \right) \\ M_3 &= \frac{1}{4(s+t)\Delta} \left(2(s+5t) \cosh^2(2s) + 4t (t^2 + 5st - 2) \cosh(2s) \right. \\ &\quad \left. + 2(s+t) (2t^2 + 8 \sinh(2s)t - 1) \right) \\ M_4 &= \frac{1}{2(s+t)\Delta} \left(4(s-t)t \cosh^2(2s) + (4(s-t)t + (s-7t) \sinh(2s)) \cosh(2s) \right. \\ &\quad \left. + 8t(t-s) + (-2t^3 + 14st^2 + 7t-s) \sinh(2s) \right) \\ M_5 &= \frac{1}{2\Delta} \left((\cosh(2s) - 1) (2t^2 + \cosh(2s) - 1) \right), \\ M_6 &= \frac{1}{2\Delta} \left((-2t^2 + \cosh(2s) - 1) \sinh(2s) \right), \end{aligned} \quad (100)$$

with Δ given by Eq.(24). These are the same expressions as in our paper [29] dealing with the zero temperature case.

The new temperature induced contributions from adding (71) and (83) read

$$\begin{aligned} M_7 &= \frac{iN}{2(s+t)Tp_4} \frac{4(s-t) \cosh(2s)}{s+t}, \\ M_8 &= \frac{iN}{2(s+t)Tp_4} \frac{4 \cosh^2(2s) - 4(2t^2 + 1) \cosh(2s)}{2\Delta}, \\ M_9 &= \frac{iN}{2(s+t)T} 4 \frac{t \cosh^2(2s) + (t + \sinh(2s)) \cosh(2s) - 2t + (2t^2 - 1) \sinh(2s)}{\Delta}. \end{aligned} \quad (101)$$

Furthermore it holds $M_{10a} = M_{10b} = 0$ and $M_{(*)} = -M_9$. Here a comment is in order. The vanishing of M_{10a} and M_{10b} is a result of the calculations done here in

one loop order. At the moment it is not known whether there is some symmetry behind and whether this persists in higher loops. Similar remarks apply to the relation between M_9 and $M_{(*)}$. As a result, these two form factors contribute proportional to the tensor structure

$$T_{\lambda\lambda'}^9 - T_{\lambda\lambda'}^{(*)} = u_\lambda id_{\lambda'} - id_\lambda u_{\lambda'} + iF(up) - \frac{u_\lambda u_{\lambda'}}{(up)}. \quad (102)$$

In addition to (99) with the form factors (100) and (101) we also have the contribution from the Debye mass (96) which with (10) and (8) can be written in the form,

$$\Pi^D u_\mu u_\nu. \quad (103)$$

It should be mentioned that all the contributions (99), and, in particular, this one, are valid off the shell of the Matsubara values for the external momentum, i.e., for arbitrary p_4 . In this case, of course, even the weak transversality does not hold and Eq.(103) is just the contribution on which this is realized. For $p_4 = 2\pi Tl$ with integer l , from Eq.(96) $\Pi^D = -\delta_{l,0}m_D^2(B)$ follows and, as discussed in section 2, the transversality holds.

Finally we note that, like in the case of the neutral polarization tensor in [31], the form factors of the charged one contain imaginary parts. These result from the tachyonic mode and must be treated in the same way as in [31]. An example is Eq. (98).

One obvious application of the results obtained in our investigations is the resummation of perturbation series in the field at high temperature. That can be done by means of the solution of the Schwinger-Dyson equations for two-point Green's functions, where the derived tensor structures with arbitrary form factors have to be used as the input propagators. Whether or not the spectrum of gluons becomes stable due to the gluon magnetic mass, which is generated in the field at high temperature in this so-called super-daisy resummation, is an interesting problem for future investigation.

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Appendix

In this appendix we collect formulas which are necessary to restore the form factors from the expressions we are getting in the main text for the parts of the polarization tensor under the signs of the parametric integrals. In order not to

overload the notations we start from the $T = 0$ case. A typical expression which appears from the calculation of the polarization tensor in the subsections 2.1 and 2.2 has the form

$$M = (Xp)_\lambda (Yp)_{\lambda'} + (Y^\top p)_\lambda (X^\top p)_{\lambda'} + W_{\lambda\lambda'}, \quad (104)$$

with

$$\begin{aligned} X &= r \delta^\parallel + \alpha iF + \beta \delta^\perp, \\ Y &= s \delta^\parallel + \gamma iF + \delta \delta^\perp, \\ W &= a \delta^\parallel + c iF + b \delta^\perp. \end{aligned} \quad (105)$$

We note that the parameters in (105) has to fulfill certain relations if M shall obey the weak transversality condition (1). Irrespective of that, the first four form factors can be restored. Using (4) we rewrite (104) in the form,

$$\begin{aligned} M &= -2rsT^{(1)} - 2(\alpha\gamma + \beta\delta)T^{(2)} - (r\delta + s\beta)T^{(3)} + (r\gamma - s\alpha)T^{(4)} \\ &\quad + A \delta^\parallel + C iF + B \delta^\perp \end{aligned} \quad (106)$$

with

$$\begin{aligned} A &= 2rsl^2 + (r\delta + s\beta)h^2 + r\gamma - s\alpha + a, \\ B &= 2\beta\delta h^2 + (r\delta + s\beta)l^2 + \alpha\delta - \beta\gamma + b, \\ C &= (s\alpha - r\gamma)l^2 + (\alpha\delta - \beta\gamma)h^2 + 2\alpha\gamma + 4\beta\delta + c. \end{aligned} \quad (107)$$

From (106), the first four form factors read

$$\begin{aligned} M_1 &= -2rs, & M_2 &= -2(\alpha\gamma + \beta\delta), \\ M_3 &= -(r\delta + s\beta), & M_4 &= r\gamma - s\alpha. \end{aligned} \quad (108)$$

In the main text, the parts of the polarization tensor appear (partially after integrating by parts) in specific forms matching one of the following representations of W ,

$$\begin{aligned} W_1 &= -2Y_{\lambda\lambda'}^\top(\widetilde{pXp}), \\ W_2 &= -2X_{\lambda\lambda'}(\widetilde{pY^\top p}), \\ W_3 &= \frac{1}{2}(W_1 + W_2). \end{aligned} \quad (109)$$

First we consider W_1 and use $(\widetilde{pXp}) = rl^2 - \alpha + \beta(h^2 + 2iF)$ to obtain

$$\begin{aligned} W_1 &= -2(s\delta^\parallel - \gamma iF + \delta \delta^\perp)(rl^2 - \alpha + \beta(h^2 + 2iF)), \\ &= -2s((rl^2 - \alpha + \beta h^2)\delta^\parallel + (-\gamma(rl^2 - \alpha + \beta h^2) + \beta\delta)iF \\ &\quad + (\delta(rl^2 - \alpha + \beta h^2) - \beta\gamma)\delta^\perp), \end{aligned} \quad (110)$$

from which we identify the corresponding expressions for a , b and c in the second line in Eq.(105). These we insert into A , B and C in (107),

$$\begin{aligned} A &= (r\delta - s\beta)h^2 + r\gamma + s\alpha , \\ B &= -(r\delta - s\beta)l^2 + 3(\alpha\delta + \beta\gamma) , \\ C &= (r\gamma + s\alpha)l^2 + (\alpha\delta + \beta\gamma)h^2 . \end{aligned} \quad (111)$$

Comparison with (4) shows that these contributions collect just into form factors such that M , (104), with W_1 from (110) is weak transversal. From (111) and (4) the remaining form factors follow,

$$M_5 = r\delta - s\beta, \quad M_{6a} = r\gamma + s\alpha, \quad M_{6b} = \alpha\delta + \beta\gamma. \quad (112)$$

The same procedure can be repeated for W_2 . The result is just that the form factors M_5 , M_{6a} and M_{6b} are the same but with reversed sign. Finally, for W_3 , they just compensate each other. Collected together, these results read

| | M_5 | M_{6a} | M_{6b} | |
|-------|-----------------------|------------------------|---------------------------------|---|
| W_1 | $r\delta - s\beta$ | $r\gamma + s\alpha$ | $\alpha\delta + \beta\gamma$ | |
| W_2 | $-(r\delta - s\beta)$ | $-(r\gamma + s\alpha)$ | $-(\alpha\delta + \beta\gamma)$ | |
| W_3 | 0 | 0 | 0 | . |

(113)

Also we need the special case when $Y^\top = X$. In that case there is only one expression for W and we have to consider (this is M from (104) divided by two)

$$M = (Xp)_\lambda (X^\top p)_{\lambda'} + W_{\lambda\lambda'}, \quad (114)$$

with

$$W = -X_{\lambda\lambda'}(p\widetilde{X}p), \quad (115)$$

In this case the nonzero form factors are

$$M_1 = -r^2, \quad M_2 = a l^2 - \beta^2, \quad M_3 = -r\beta, \quad M_4 = -r\alpha. \quad (116)$$

Now we generalize these formulas to finite temperature. The basic expressions which appear in the subsections above have the form

$$M = (Xp - \mu u)_\lambda (Yp - \nu u)_{\lambda'} + (Y^\top p - \nu u)_\lambda (X^\top p - \mu u)_{\lambda'} + W_{\lambda\lambda'}, \quad (117)$$

Here μ and ν are numbers and for W again 3 types of expressions appear,

$$\begin{aligned} W_1 &= -2 \left(Y_{\lambda\lambda'}^\top - \nu \frac{u_\lambda u_{\lambda'}}{(up)} \right) ((p\widetilde{X}p) - \mu(up)) , \\ W_2 &= -2 \left(X_{\lambda\lambda'} - \mu \frac{u_\lambda u_{\lambda'}}{(up)} \right) ((p\widetilde{Y}^\top p) - \nu(up)) , \\ W_3 &= \frac{1}{2} (W_1 + W_2) . \end{aligned} \quad (118)$$

Let us consider the contributions ΔW which are new as compared to Eq.(104). We start from the easiest part, namely that proportional to $\mu\nu$. As easily can be seen from (117) and (118) they cancel so that only contributions linear in μ and linear in ν remain. For W_1 these read

$$\begin{aligned}\Delta W_1 = & -\nu \left[\frac{r}{(up)} (T^{(7)} + T^{(10a)}) + \frac{\beta}{(up)} (T^{(8)} + T^{(10b)}) - \alpha (T^{(9)} - 2T^{(*)}) \right] \\ & -\mu \left[\frac{s}{(up)} (T^{(7)} - T^{(10a)}) + \frac{\delta}{(up)} (T^{(8)} - T^{(10b)}) + \gamma T^{(9)} \right].\end{aligned}\quad (119)$$

Here we used in addition to (7) the following formulas which are easy to verify,

$$\begin{aligned}T^{(7)} + T^{(10a)} &= (u_\lambda l_{\lambda'} + l_\lambda u_{\lambda'}) (up) - 2u_\lambda u_{\lambda'} l^2, \\ T^{(7)} - T^{(10a)} &= (u_\lambda l_{\lambda'} + l_\lambda u_{\lambda'}) (up) - 2\delta_{\lambda\lambda'}^\parallel (up)^2, \\ T^{(8)} + T^{(10b)} &= (u_\lambda h_{\lambda'} + h_\lambda u_{\lambda'}) (up) - 2u_\lambda u_{\lambda'} h^2, \\ T^{(7)} - T^{(10b)} &= (u_\lambda h_{\lambda'} + h_\lambda u_{\lambda'}) (up) - 2\delta_{\lambda\lambda'}^\perp (up)^2\end{aligned}\quad (120)$$

and

$$T^{(9)} - 2T^{(*)} = u_\lambda id_{\lambda'} - id_\lambda u_{\lambda'} - \frac{2}{(up)} u_\lambda u_{\lambda'} \quad (121)$$

with

$$\begin{aligned}T^{(*)} &= \frac{1}{(up)} (u_\lambda u_{\lambda'} + (up)^2 iF) \\ &= \frac{1}{(up)} \frac{1}{p^2 + 2iF} ((up)^2 T^{(6)} - T^{(10)}) .\end{aligned}\quad (122)$$

In a similar way we get

$$\begin{aligned}\Delta W_2 = & -\nu \left[\frac{r}{(up)} (T^{(7)} - T^{(10a)}) + \frac{\beta}{(up)} (T^{(8)} - T^{(10b)}) - \alpha T^{(9)} \right] \\ & -\mu \left[\frac{s}{(up)} (T^{(7)} + T^{(10a)}) + \frac{\delta}{(up)} (T^{(8)} + T^{(10b)}) + \gamma (T^{(9)} - 2T^{(*)}) \right].\end{aligned}\quad (123)$$

Taking the half sum of both the expression

$$\begin{aligned}\Delta W_3 = & -\nu \left[\frac{r}{(up)} T^{(7)} + \frac{\beta}{(up)} T^{(8)} - \alpha (T^{(9)} - T^{(*)}) \right] \\ & -\mu \left[\frac{s}{(up)} T^{(7)} + \frac{\delta}{(up)} T^{(8)} + \gamma (T^{(9)} - T^{(*)}) \right]\end{aligned}\quad (124)$$

emerges.

Finally we note the special case with $Y = X^\top$, $\mu = \nu$ for which we have

$$M = (Xp - \mu u)_\lambda \left(X^\top p - \mu u \right)_{\lambda'} + W_{\lambda\lambda'}, \quad (125)$$

with

$$W = - \left(X_{\lambda\lambda'} - \mu \frac{u_\mu u_\nu}{(up)} \right) \left((p\widetilde{X}p) - \mu(up) \right). \quad (126)$$

We obtain in addition to (116)

$$\Delta W = -\mu \left[\frac{r}{(up)} T^{(7)} + \frac{\beta}{(up)} T^{(8)} - \alpha \left(T^{(9)} - T^{(*)} \right) \right]. \quad (127)$$

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